Algebraic invariants of five qubits

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The Hilbert series of the algebra of polynomial invariants of five qubits pure states is obtained, and the simplest invariants are computed.

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INTRODUCTION

Quantifying entanglement in multipartite systems is a fundamental issue in Quantum Information Theory. However, for systems with more than two parts, very little is know in this respect. A few useful entanglement measures for pure states of 3 or 4 qubits have been investigated [1, 2, 3], but one is still far from a complete understanding. Furthermore, for system of up to 4 qubits, a complete classification of entanglement patterns and of corresponding invariants under local filtering operations (also known as SLOCC, Stochastic Local Operations assisted by Classical Communication) is know [4, 5]. Klyachko [6, 7] proposed to associate entanglement (of pure states) in a k-partite system (or perhaps , one should say "pure k-partite" entanglement) with the mathematical notion of semi-stability, borrowed from geometric invariant theory, which means that at least one SLOCC invariant is non zero. For such states, the absolute values of these invariants provide some kind of entanglement measure. However, even for system of k qubits, the complexity of these invariants grows very rapidly with the number of parts. For $k = 2$, they are given by simple linear algebra [8, 9]. The case $k = 3$ is already nontrivial but appears in the physics literature in [10] and boils down to a mathematical result which was known by 1880 [11]. The case $k = 4$ is quite recent [5], and to the best of our knowledge, nothing was known for 5-qubit systems[22].

Our main result is a closed expression of the Hilbert series of the algebra of SLOCC invariants of pure 5-qubit states. This result, which determines the number of linearly independent homogeneous invariants in any degree, was obtained through intensive symbolic computations relying on a very recent algorithm for multivariate residue calculations. We point out a few properties which can be read off from the series, and determine the simplest invariants, which are of degree 4 and 6 in the component of the states.

HILBERT SERIES

Denote by $V = \mathbb{C}^2$ the local Hilbert space of a two state particle. The state space of a five particule system is $\mathcal{H} = V^{\otimes 5}$, which will be regarded as the natural representation of the group of invertible local filtering operations, also known as reversible stochastic local quantum operations assisted by classical communication

$$G = G_{\mathrm{SLOCC}} = \mathrm{SL}(2, \mathbb{C})^{\times 5},$$

that is, the group of 5-tuples of complex unimodular 2×2 matrices. We will denote by

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4,i_5=0}^{1} A_{i_1i_2i_3i_4i_5}|i_1\rangle|i_2\rangle|i_3\rangle|i_4\rangle|i_5\rangle$$

a state of the system. An element $\mathbf{g} = ({}^k g_i^j)$ of G maps $|\Psi\rangle$ to the state

$$|\Psi'\rangle = \mathbf{g}|\Psi\rangle$$

whose components are given by

$$A'_{i_1i_2i_3i_4i_5} = \sum_{\mathbf{j}} {}^1g_{i_1}^{j_1}\,{}^2g_{i_2}^{j_2}\,{}^3g_{i_3}^{j_3}\,{}^4g_{i_4}^{j_4}\,{}^5g_{i_5}^{j_5} A_{j_1j_2j_3j_4j_5} \qquad (1)$$

We are interested in the dimension of the space $\mathcal{I}_d$ of all G-invariant homogeneous polynomials of degree $d = 2m$ ($\mathcal{I}_d = 0$ for odd d) in the 32 variables $A_{i_1i_2i_3i_4i_5}$.

It is known that it is equal to the multiplicity of the trivial character of the symmetric group $\mathfrak{S}_{2m}$ in the fifth power of its irreducible character labeled by the partition $[m, m]$

$$\dim \mathcal{I}_d = \langle \chi^{2m} | (\chi^{mm})^5 \rangle. \qquad (2)$$

The generating function of these numbers

$$h(t) = \sum_{d\geq 0} \dim \mathcal{I}_d\, t^d \qquad (3)$$

is called the Hilbert series of the algebra $\mathcal{I} = \bigoplus_d \mathcal{I}_d$. Standard manipulations with symmetric functions allow to express it as a multidimensional residue:

$$h(t) = \oint \frac{du_1}{2\pi\imath u_1} \cdots \oint \frac{du_5}{2\pi\imath u_5} \frac{A(\mathbf{u})}{B(\mathbf{u}; t)} \qquad (4)$$

where the contours are small circles around the origin,

$$A(\mathbf{u}) = \prod_{i=1}^{5} \left(1 + 1/u_i^2\right) \qquad (5)$$

n	a_n	n	a_n	n	a_n	n	a_n
0	1	30	24659	54	225699	78	9664
8	16	32	36611	56	214238	80	5604
10	9	34	52409	58	195358	82	3024
12	82	36	71847	60	172742	84	1659
14	145	38	95014	62	146849	86	770
16	383	40	119947	64	119947	88	383
18	770	42	14849	66	95014	90	145
20	1659	44	172742	68	71847	92	82
22	3024	46	195358	70	52409	94	9
24	5604	48	214238	72	36611	96	16
26	9664	50	225699	74	24659	104	1
28	15594	52	229752	76	15594		

TABLE I: Coefficients of $P(t)$

and

$$B(\mathbf{u};t) = \prod_{a_i=\pm 1} (1 - t\, u_1^{a_1} u_2^{a_2} u_3^{a_3} u_4^{a_4} u_5^{a_5}) \qquad (6)$$

Such multidimensional residues are notoriously difficult to evaluate. After trying various approaches, we eventually succeded by means of a recent algorithm due to Guoce Xin [12], in a `Maple` implementation. The result can be cast in the form

$$h(t) = \frac{P(t)}{Q(t)} \qquad (7)$$

where $P(t)$ is an even polynomial of degree 104 with non negative integer coefficients a_n

$$P(t) = \sum_{k=0}^{52} a_{2k} t^{2k}$$

given in table I, and

$$Q(t) = (1-t^4)^5(1-t^6)(1-t^8)^5(1-t^{10})(1-t^{12})^5.$$

On this expression, it is clear that a complete description of the algebra of G-invariant polynomials by generators and relations is out of reach of any computer system. Nevertheless, inspection of the Hilbert series suggests the following kind of structure for this algebra. We know, since $\dim \mathcal{H} - \dim G = 2^5 - 3 \times 5 = 17$, that there must exist a set of 17 algebraically independent invariants. The denominator of the series, which is precisely a product of 17 factors, makes it plausible that these invariants can be choosen as five polynomials of degree 4 (to be denoted by D_xD_y, D_z, D_t, D_u), one polynomial of degree 6 (F), five polynomials of degree 8 ($H_1, H_2 \ldots, H_5$), one polynomial of degree 10 (J) and five polynomials of degree 12 ($L_1, \ldots, L_5$). These 17 polynomials are called the primary invariants.

The numerator should then describe the secondary invariants, that is, a set of 3014400 homogeneous polynomials (1 of degree 0 , 16 of degree 8, 9 of degree 10, 82 of degree 12 etc) such that any invariant polynomial can be uniquely expressed as a linear combination of secondary invariants, the coefficients being themselves polynomials in the primary invariants.

This picture, which is the simplest kind of description to be expected, is far too complex for physical applications. The best that can be done is to use the Hilbert series as a guide for finding explicitly a small set of reasonably simple invariants, in particular, the primary invariants of lowest degrees. We have computed the first primary invariants, those of degree 4 and 6, using methods from Classical Invariant Theory (see below).

THE SIMPLEST INVARIANTS

Transvectants and Cayley's Omega process

In order to apply the formalism of Classical Invariant Theory, a state $|\Psi\rangle$ will be interpreted as a quintilinear form on $\mathbb{C}^2$ (called the ground form)

$$f := \sum_{i_1,i_2,i_3,i_4 i_5=0}^{1} A_{i_1 i_2 i_3 i_4 i_5} x_{i_1} y_{i_2} z_{i_3} t_{i_4} u_{i_5}$$

A covariant of f is a G-invariant polynomial in the coefficients $A_{i_1 i_2 i_3 i_4 i_5}$ and the variables x_i, y_i, z_i, t_i and u_i. A complete set of covariants can be in principle computed from the ground form by means of the so-called Omega process (see [13, 14] for notations). Cayley's Omega process consists in applying iteratively differential operators called transvections and defined by

$$(P,Q)^{\epsilon_1 \cdots \epsilon_5} \;=\; \mathrm{tr}\ \Omega_x^{\epsilon_1} \cdots \Omega_u^{\epsilon_5} P(x', \ldots, u') Q(x'', \ldots, u'')$$

where

$$\Omega_x = \det \begin{vmatrix} \frac{\partial}{\partial x'_1} & \frac{\partial}{\partial x'_2} \\ \frac{\partial}{\partial x''_1} & \frac{\partial}{\partial x''_2} \end{vmatrix}$$

and $\mathrm{tr} : x', x'' \to x$.

Degree 4

Regarding x as a parameter, write f as a quadrilinear binary form in the variables y_i, z_i, t_i and u_i

$$f = \sum A^x_{i_1 i_2 i_3 i_4} y_{i_1} z_{i_2} t_{i_3} u_{i_4}$$

It is known that such a quadrilinear form admits an invariant of degree 2 (called Cayley's hyperdeterminant

[15, 16, 17]) which is a quadratic binary form b_x in the variables $x = (x_1, x_2)$. Hence, taking the discriminant of b_x one obtains an invariant D_x of degree 4. We repeat this operation for the other binary variables and we obtain four other invariants D_y, D_z, D_t and D_u. Evaluating the appropriate Jacobians with a computer algebra system gives the algebraic independance of the five invariants.

Degree 6

We obtain the primary invariant of degree 6 by a succession of transvections. First, we compute a triquadratic covariant of degree 2

$$B_{22020} = (f, f)^{00101}.$$

This covariant allows to construct a cubico-quadrilinear covariant of degree 3

$$C_{31111} = (B_{22020}, f)^{01010}$$

which gives a triquadratic polynomial of degree 4

$$D_{22200} = (C_{31111}, f)^{10011}.$$

Hence, one obtains a quintilinear covariant of degree 5

$$E_{11111} = (D_{22200}, f)^{11100}.$$

Finally, we find the invariant of degree 6

$$F = (E_{11111}, f)^{11111}.$$

By computing the Jacobian, one finds that F is algebraically independent of $D_x, \ldots, D_u$.

CONCLUSION

From the Hilbert series, it appears that the algebra of polynomial invariants of a five qubit system has a very high complexity. Furthermore, as is already the case with smaller systems [4, 5, 17], the knowledge of the invariants is not sufficient to classify entanglement patterns. In the case of four qubits or three qutrits, this classification can be achived due to hidden symmetries which have their roots in very subtle aspects of the theory of semi-simple Lie algebras (Vinberg's theory [18]). However, such symmetries are absent in the case of 5 qubits. Then, the only known general approach for classifying orbits (entanglement patterns) requires the computation of the algebra of covariants, which is already almost intractable in the case of four qubits. It has 170 generators, which have been found [5], but the description of their algebraic relations (syzygies) is definitely out of reach. However, a closer look at the 4-qubit system, reveals that the classification of Verstraete et al [4, 19]. can be reproduced by means of only a small set of covariants. We hope that our results will allow the identification and the calculation of such a small set of invariants and covariants, sufficient to separate the physically relevant entanglement patterns, which are probably not so numerous. To illustrate this principle, let us consider a result of Osterloh and Siewert [20]. Having introduced a notion of *filter* which can be used to separate SLOCC orbits in the same way as covariants, these authors show that the four states

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|11111\rangle + |00000\rangle)$$

$$|\Phi_2\rangle = \frac{1}{2}(|11111\rangle + |11100\rangle + |00010\rangle + |00001\rangle)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}\,|11111\rangle + |11000\rangle + |00100\rangle + |00010\rangle + |00001\rangle\right)$$

$$|\Phi_4\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}\,|11111\rangle + |10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle\right)$$

are in different orbits. As can be seen on Table II, the orbits of these states are also distinguished by our covariants.

	$\|\Phi_1\rangle$	$\|\Phi_2\rangle$	$\|\Phi_3\rangle$	$\|\Phi_4\rangle$
D_x	$\times$	$\times$	0	0
D_y	$\times$	$\times$	0	0
D_z	$\times$	0	0	0
D_t	$\times$	0	0	0
D_u	$\times$	0	0	0
F	0	0	0	0
B_x	$\times$	$\times$	$\times$	$\times$
C_{31111}	0	0	$\times$	$\times$
E_{11111}	0	$\times$	0	$\times$

TABLE II: Evaluation of SLOCC covariants for Osterloh and Siewert states ($\times$ means that the evaluation is not 0)

Finally, the investigation of entanglement measures requires an understanding of invariants under local unitary transformations (LUT) [21] . In a forthcoming paper, we will explain how to obtain LUT-invariants from SLOCC-covariants.

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[22] Just after we posted the first version of this Note, A. Osterloh and J. Siewert informed us of their independent work [20] on the five qubits problem (see Conclusion for a short discussion).